Also solved by Adnan Ali (student), A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Nikos Kalapodis, Patras, Greece; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy; Henry Ricardo, New York Math Circle, NY; Albert Stadler, Herrliberg, Switzerland; Toshihiro Shimizu, Kawaskaki, Japan; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Nicusor Zlota "Traian Vuia" Technical College, Focansi,Romania, and the proposer

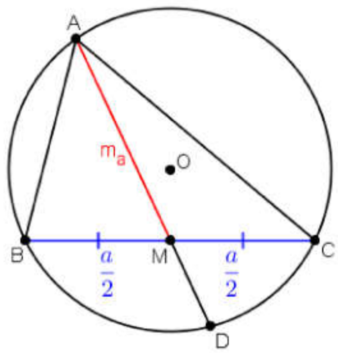
5400: *Proposed by Arkady Alt, San Jose, CA*

Prove the inequality

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\le 12\,(2R-3r),$$

where a,b,c and m_a,m_b,m_c are respectively sides and medians of $\triangle ABC$, with circumradius R and inradius r.

Solution 1 by Nikos Kalapodis, Patras, Greece



Let the median $AM=m_a$ intersects the circumcircle of triangle ABC at D.

Then by the intersecting chords theorem we have
$AM\cdot MD=MB\cdot MC$ or $AM\cdot(AD-AM)=MB\cdot MC$.

It follows that $m_a\cdot AD-m_a^2=\dfrac{a^2}{4}$ i.e. $\dfrac{a^2}{m_a}=4AD-4m_a$.

By the obvious inequality $AD\le 2R$ we obtain that $\dfrac{a^2}{m_a}\le 8R-4m_a$ (1).

Taking into account the other two similar inequalities we have

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\le 24R-4(m_a+m_b+m_c)\quad (2).$$

Inequality (1) can be rewritten as $m_a\ge\dfrac{a^2+4m_a^2}{8R}$. Adding the other two similar inequalities and using the following well-known identities

$m_a^2+m_b^2+m_c^2=\dfrac{3}{4}(a^2+b^2+c^2)$, $bc=2Rh_a$, $\dfrac{1}{h_a}+\dfrac{1}{h_b}+\dfrac{1}{h_c}=\dfrac{1}{r}$ we get that

$$
\begin{aligned}
m_a+m_b+m_c \geq \frac{a^2+b^2+c^2+4(m_a^2+m_b^2+m_c^2)}{8R} &= \frac{a^2+b^2+c^2}{2R} \geq \frac{bc+ca+ab}{2R}\\
&= h_a+h_b+h_c \geq \frac{9}{\dfrac{1}{h_a}+\dfrac{1}{h_b}+\dfrac{1}{h_c}}\\
&= \frac{9}{\dfrac{1}{r}} = 9r, \quad \text{i.e.} \quad m_a+m_b+m_c \geq 9r \quad (3).
\end{aligned}
$$

Combining (2) and (3) we have

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c} \leq 24R-4(m_a+m_b+m_c) \leq 24R-4\cdot 9r = 12(2R-3r).$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Let A', B', C' and $A", B", C"$ be respectively the midpoints of the sides BC, CA, AB and the intersections of the medians AA', BB', CC' with the circumcircle of $\triangle ABC$ and let us denote h_a, h_b, h_c the heights and $n_a = A'A'', n_b = B'B'', n_c = C'C''$

Taking into account that the absolute value of the power of A' with respect to the circumcircle of $\triangle ABC$ is $A'B\cdot A'C$ and also $A'A\cdot A'A''$, that is $\frac{a}{2}\cdot\frac{a}{2} = m_a\cdot n_a$ or equivalently $\frac{a^2}{m_a} = 4n_a$.

Since $m_a+n_a \leq 2R$ ($AA^{'}$ is a chord of the circumcircle whose diameter is $2R$) and $h_a \leq m_a$ (the height is the minimum distance from the vertex to its opposite side), we conclude that $na_a \leq 2R-m_a \leq 2R-h_a$.

Thus $\frac{a^2}{m_a} \leq 4(2R-h_a)$ and analogously $\frac{b^2}{m_b} \leq 4(2R-h_b)$ and $\frac{c^2}{m_c} \leq 4(2R-h_c)$ so

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c} \leq 12\left(2R-\frac{1}{3}(h_a+h_b+h_c)\right).$$

The result follows from $h_a+h_b+h_c \geq 9r$, with equality iff $\triangle ABC$ is equilateral which is equality 6.8 from page 61 in the book *Geometric inequalities* by O. Bottema, R. Ž. Djordjević, R.R. Janić, D.S. Mitrinović and P.M Vasić, Wolters Noordhoff, Groningen, 1969.

Equality is attained iff $m_a+n_a = 2R,\ h_a = m_a$ and $\mathrm{h}_a+h_b+h_c = 9r$ and cyclically, that is, iff $\triangle ABC$ is an equilateral triangle.

Solution 3 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

Using the inequality $m_a \geq \frac{b^2+c^2}{4R}$, we obtain

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c} \quad \leq \quad \sum\frac{4Ra^2}{b^2+c^2}$$

$$\frac{2(2R-3r)}{R}-\sum\frac{a^2}{b^2+c^2} \geq 0 \quad \Longleftrightarrow \quad \frac{3(2R-3r)}{R}-\frac{3}{2} \geq 0 \Longrightarrow R \geq 2r, \text{ which is true.}$$